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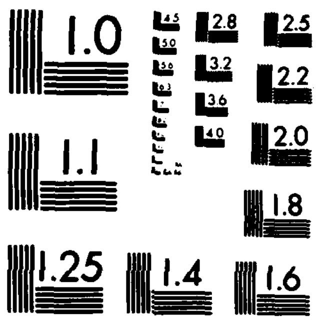

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ON THE SUPERLINEAR AMBROSETTI-PRODI PROBLEM

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ABSTRACT

This paper establishes the existence of two solutions for some problems of the Ambrosetti-Prodi type. The following result is a sample of the results. Let $g : R \to R$ be a C^1-function such that $\lim\limits_{s\to-\infty} g'(s) < \lambda_1 < \lim\limits_{s\to+\infty} g'(s)$, where λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions, and the limits could be infinite. Suppose that g behaves at $+\infty$ like u^p, $1 < p < (N + 2)/(N - 2)$ or even like $u^{(N+2)/(N-2)}/\ln u$, where $N > 3$. Let Ω be a bounded smooth domain in R^N and let ϕ be the first eigenfunction corresponding to λ_1, which is > 0 in Ω. Then given any $h \in C^\alpha(\bar{\Omega})$ such that $\int h\phi = 0$, there is $t_0 \in R$ such that the Dirichlet problem

$$-\Delta u = g(u) + t\phi + h, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

has at least two $C^{2,\alpha}$ solutions for each $t < t_0$. The author uses the method of monotone iterations to obtain the first solution and a variational argument to get the second. The variational solution is subsequently regularized.

AMS (MOS) Subject Classifications: 35J65, 47H07, 58E30

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ON THE SUPERLINEAR AMBROSETTI-PRODI PROBLEM

Djairo G. de Figueiredo*

INTRODUCTION. Let Ω be a smooth bounded domain in R^N. We consider the semilinear elliptic boundary value problem

$$(1) \qquad -\Delta u = g(x,u) + f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega ,$$

where $f(x)$ is some given C^α-function in Ω, and the nonlinearity g satisfies the smoothness condition below, besides other conditions that will be timely introduced as we proceed:

$$(2) \qquad g : \bar{\Omega} \times R \to R \quad \textit{is a } C^1\textit{-function.}$$

Problem (1) is said to be of the Ambrosetti-Prodi type if

$$(3a) \quad \limsup_{s\to-\infty} \frac{g(x,s)}{s} < \lambda_1 , \quad \text{and} \qquad (3b) \quad \lambda_1 < \liminf_{s\to+\infty} \frac{g(x,s)}{s} ,$$

where the inequalities hold uniformly in Ω, and the limits could assume value $-\infty$ or $+\infty$, respectively, on the whole of Ω or on subsets of positive measure. Here λ_1 denotes the first eigenvalue of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u = 0$ on $\partial\Omega$. Let ϕ be the eigenfunction corresponding to λ_1 which is >0 in Ω and $\int \phi^2 = 1$. And let W be the subspace of $C^\alpha(\bar{\Omega})$ generated by ϕ, and $W^\perp = \{u \in C^\alpha(\bar{\Omega}) : \int u\phi = 0\}$. Consequently any $f \in C^\alpha(\bar{\Omega})$ can be uniquely written as $f = t\phi + h$, with $t \in R$ and $h \in W^\perp$. Using this decomposition we shall look at the parametrized form of (1):

$$(1_t) \qquad -\Delta u = g(x,u) + t\phi + h \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega .$$

Conditions (3a) and (3b) are implied by the more restrictive conditions (4a) and (4b), respectively:

$$(4a) \quad \limsup_{s\to-\infty} g_s'(x,s) < \lambda_1 , \qquad (4b) \quad \lambda_1 < \liminf_{s\to+\infty} g_s'(x,s) ,$$

where g_s' denotes the partial derivative of g with respect to s.

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Viewing the future use of variational methods we suppose

$$(5)\qquad \lim_{s\to+\infty} \frac{g(x,s)}{s^\sigma} = 0\ , \qquad 1 < \sigma < \frac{N+2}{N-2} \ \text{ if } \ N \geq 3\ ,$$
$$1 < \sigma < \infty \qquad \text{if } \ N = 2$$

No assumption on the growth of g as $s \to -\infty$ will be necessary. Also by force of some desirable Palais-Smale condition for a functional associated with our problem (1) we found it necessary to assume the following technical condition, but only in the case $N \geq 3$ and $(N + 1)/(N - 1) < \sigma < (N + 2)/(N - 2)$:

$$(6)\qquad \liminf_{s\to+\infty} \frac{sg(x,s) - \theta G(x,s)}{s^2 g(x,s)^{2/(N+1)}} > 0\ , \quad \text{for some} \ \ \theta > 2\ ,$$

where $G(x,s) = \int_0^s g(x,\xi)d\xi$.

<u>Theorem 1</u>. <u>Assume</u> (2), (3b), (4a), (5) <u>and</u> (6). <u>Then for each given</u> $h \in N^\perp$ <u>there exists</u> $t_0 \in R$ <u>such that</u> (1_t) <u>has at least two</u> $C^{2,\alpha}$ <u>solutions for</u> $t < t_0$.

The first result of this sort was proved by Ambrosetti-Prodi [1], under much stronger assumptions, but yielding a sharper conclusion. In their work as well as in the subsequent work of other authors, e.g., Berger-Podolak [2], Amann-Hess [3], Fučik [4], the nonlinearity g is assumed to have linear growth. Kazdan-Warner [5] relaxed this condition but obtained only one solution. Dancer [6] obtains a second solution, provided $\sigma \leq (N + 1)/(N - 1)$ in (5). He uses a topological degree argument, and a crucial point in his proof is obtaining a priori estimates on the solutions of (1_t). Under his restriction on σ these estimates are readily obtained using a technique due to Brézis-Turner [7]. For nonlinearities g growing faster than $(N + 1)/(N - 1)$ this is not a simple matter, and as far as we know estimates are not available yet. We found it easier to obtain a second solution in this case using a variational method. Here also there is a delicate point which is the establishment of the Palais-Smale condition for some associated functional. We required a technical condition (6), which we believe to be not too restrictive. For example, pure powers like $g(x,s) = a(x)s^p$ for large s and $1 < p < (N+2)/(N-2)$ do satisfy (6). More generally the condition introduced by Ambrosetti-Rabinowitz [8] in similar situations also gives (6). For the early history of the Ambrosetti-Prodi problem see the survey paper [9]. We should also mention the work of Berestycki-Lions [17] on the superlinear Ambrosetti-Prodi with convex nonlinearities g.

We mention that the steps in the proof of Theorem 1 will make clear that a theorem of Ambrosetti-Rabinowitz [8] on the existence of positive solutions for some semilinear elliptic equations can be slightly extended. For completeness let us state it in case of the Laplacian.

Theorem 2 [Ambrosetti-Rabinowitz]. Let $g : \Omega \times R^+ \to R^+$ be a Carathéodory function satisfying conditions (3b), (5) and (6) above as well as $g(x,0) = 0$ and

$$\limsup_{s \to 0} \frac{g(x,s)}{s} < \lambda_1$$

Then the boundary value problem

$$-\Delta u = g(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

has a nontrivial positive solution u in H_0^1. If g is C^α then $u \in C^{2,\alpha}(\bar{\Omega})$.

The above theorem should be compared with Theorem 2.3 in de Figueiredo-Lions-Nussbaum [10]. Although Theorem 2 above applies to more general second order elliptic operators and nonlinearities g depending on x, its technical condition (6) seems more restrictive than the requirements made in Theorem 2.3.

The paper is divided in three parts. In the first section we prove that there exists a $t_2 \in R$ such that for all $t < t_2$, problem (1_t) has a minimal negative solution $u_t \in C^{2,\alpha}(\bar{\Omega})$ such that the eigenvalue problem

$$-\Delta v - g_s'(x,u_t)v = \mu v \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega ,$$

has a positive first eigenvalue μ_1. In the second part we take an appropriate truncation of the nonlinearity g and construct a functional whose critical points are precisely the weak solutions of (1_t). It is then shown that the minimal solution u_t found in the previous section is a local minimum of this functional. Under our assumptions the functional is shown to satisfy the Palais-Smale condition. The mountain pass theorem of [8] is applied to get a second solution of (1_t) in H_0^1. Its regularity in the case $\sigma < N + 2/N - 2$ is seen by the standard way using a bootstrap argument. The case $\sigma = (N + 2)/(N - 2)$ requires a great deal more of work. We prove it in Section 3 using an argument due to Brézis-Kato [11].

The novelty of the present work is the treatment of superlinear nonlinearities g with growth at infinite "touching" the critical exponent $(N + 2)/(N - 2)$, in the sense of condition (5).

The author thanks Professor Antonio Ambrosetti for his hospitality at the Scuola Internazionale Superiore di Studi Avanzati (Trieste-Italia), where part of this work was done.

1. **EXISTENCE OF A MINIMAL NEGATIVE SOLUTION**. Since some of the assertions made in the present section are true under weaker assumptions than the ones placed in Theorem 1, we shall state them separately. Also this will make clear the role of (4a) which we believe it could be replaced by (3a), as well as the role of (6) which eventually could be either weakened or dropped altogether.

Lemma 3. Assume (3a) and (3b). Then there exists a number τ, independent of $h \in N^{\perp}$, such that (1_t) has no solution for $t > \tau$.

Proof. It follows from (3a) and (3b) that there are numbers $C > 0$ and $\underline{\mu} < \lambda_1 < \bar{\mu}$ such that

$$g(x,s) \geq \underline{\mu}s - C \quad \text{and} \quad g(x,s) \geq \bar{\mu}s - C \tag{7}$$

for all $x \in \bar{\Omega}$ and all $s \in R$. Take the inner product of (1_t) with ϕ, integrate by parts, and estimate $\int g(x,u)\phi$ using (7). This will give an upper bound on the values of t for which (1_t) has a solution. □

In this section we use the method of monotone iteration to get a minimal solution of (1_t). We say that $\underline{u} \in C^{2,\alpha}(\bar{\Omega})$ is a subsolution of (1_t) if

$$-\Delta\underline{u} \leq g(x,\underline{u}) + t\phi + h \quad \text{in} \quad \Omega, \quad u \leq 0 \quad \text{on} \quad \partial\Omega . \tag{8}$$

A supersolution is defined likewise by changing in (8) the inequalities $\leq$ for $\geq$. It is well known, see for instance [12], [13], that if (1_t) has a subsolution $\underline{u}$ and a suspersolution $\bar{u}$, such that $\underline{u} \leq \bar{u}$, then (1_t) has a solution u_t such that $\underline{u} \leq u_t \leq \bar{u}$ and if u is any other solution with $\underline{u} \leq u \leq \bar{u}$ then $u_t \leq u$. This solution is obtained by an iterative method.

Lemma 4. Assume (3a) and (3b). Given $h \in N^{\perp}$ and a compact interval $[a,b]$, there is a function $w \in C^{2,\alpha}(\bar{\Omega})$ with $w = 0$ on $\partial\Omega$ such that: (i) w is a subsolution of (1_t) for each $t \in [a,b]$, (ii) $w \leq v$ for all v which are supersolutions of (1_t) with $t \in [a,b]$. In particular, w bounds from below all solutions of (1_t) for $t \in [a,b]$.

Proof. The unique solution w of the linear problem

$$-\Delta w = \underline{\mu} w - C + a\phi + h \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

where $\underline{\mu}$ and C were introduced in (7), satisfies all the requirements of the lemma. □

Lemma 5. *Assume (3a) and (3b). Given $h \in W^{\perp}$, there is a $t_1 \in \mathbb{R}$ and a function W in $C^{2,\alpha}(\bar{\Omega})$ which is <0 in Ω, $=0$ in $\partial\Omega$ and which is a supersolution for all problems (1_t) with $t < t_1$.*

Proof. Let $\varepsilon > 0$ and $p > N$ be given. As a consequence of the Sobolev imbedding $W^{2,p}(\Omega) \subset C^{1,\alpha}(\bar{\Omega})$ and the strong maximum principle, we see that there are numbers $\alpha, K > 0$ such that

(9) $\quad -K < v - \alpha\phi < 0$ in Ω for $v \in W^{2,p}(\Omega)$ with $v = 0$ on $\partial\Omega$, and $\|v\|_{W^{2,p}} < \varepsilon$

By the L^p-regularity theory, the operator $-\Delta$ with Dirichlet boundary condition has a bounded inverse from L^p to $W^{2,p}$, i.e., there is a constant $C > 0$ such that

(10) $$\|v\|_{W^{2,p}} \leq C\|\zeta\|_{L^p} \quad \text{for } -\Delta v = \zeta \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega$$

Let

(11) $$m = \max\{|g(x,s) + h(x)| : x \in \bar{\Omega}, -K \leq s \leq \varepsilon\}$$

and take subdomains Ω_1 and Ω_2 of Ω such that $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \subset \bar{\Omega}_2 \subset \Omega$ and the Lebesgue measure of $\Omega \backslash \Omega_1$ is smaller than $[\varepsilon/Cm]^p$, where C and m have been introduced in (10) and (11) respectively. Next define a C^∞ function ζ which is 0 in Ω_1, $=m$ in $\Omega \backslash \Omega_2$ and assumes values between 0 and m in $\Omega_2 \backslash \Omega_1$. Let v be the unique solution of the Dirichlet problem $-\Delta v = \zeta$ in Ω, $v = 0$ on $\partial\Omega$. Using (10) and the choice of Ω_1 we have

(12) $$\|v\|_{W^{2,p}} \leq C\|\zeta\|_{L^p} \leq Cm|\Omega \backslash \Omega_1|^{1/p} \leq \varepsilon$$

We claim that $W = v - \alpha\phi$ is a supersolution of (1_t) provided we choose t sufficiently large negatively Indeed, if $x \in \Omega \backslash \Omega_2$

$$-\Delta W = \zeta - \alpha\lambda_1\phi = m - \alpha\lambda_1\phi \geq g(x, v - \alpha\phi) + h(x) - \alpha\lambda_1\phi$$

where the inequality follows from (9), (11) and (12). If $x \in \Omega_2$, use the fact that $\phi(x)$ is bounded away from 0 by a positive constant for $x \in \Omega_2$ and obtain $\bar{t}$ such that $\zeta(x) \geq g(x, v - \alpha\phi) + h(x) + \bar{t}\phi$. Then, for $x \in \Omega_2$

$$-\Delta W = \zeta - \alpha\lambda_1\phi \geq g(x, v - \alpha\phi) + h(x) + (\bar{t} - \alpha\lambda_1)\phi \ .$$

Thus the lemma is proved with $t_1 = \min[\bar{t} - \alpha\lambda_1, -\alpha\lambda_1]$.

<u>Remark</u>. Kazdan-Warner [5] under the assumptions of Lemma 5 have proved by a similar argument the existence of a supersolution which happens to be <u>not</u> negative in Ω. It is apparent that problem (1_t) could have a supersolution for $t > t_1$, although it is not necessarily negative.

Using the method of monotone iteration it follows

<u>Corollary 6</u>. <u>Assume</u> (3a) <u>and</u> (3b). <u>Given</u> $h \in W^{\perp}$, <u>there is a</u> $t_1 \in R$ <u>such that problem</u> (1_t) <u>has a negative solution</u> $u_t \in C^{2,\alpha}$ <u>for each</u> $t \leq t_1$. <u>Moreover</u> u_t <u>is minimal, i.e., given any solution</u> u <u>of</u> (1_t) <u>we have</u> $u_t \leq u$.

<u>Remark</u>. Existence of a negative solution in the Ambrosetti-Prodi is implicit in the work of Lazer-McKenna [14]. It was essentially noted in [9], but to our knowledge was first explicitly proved by Solimini [15]. That author [15] and Ambrosetti [16] have made a relevant use of this fact to prove multiplicity results (existence of 3 solutions) in certain Ambrosetti-Prodi problems. We remark that all these authors work with nonlinearities g having linear growth, essentially

$$g(x,u) = \mu^+ u^+ - \mu^- u^- + k(x,u)$$

where $\mu^{\pm} = \lim_{\mu \to \pm\infty} g'(u)$ and $k(x,u)$ is bounded. Their interesting proof of the existence of a negative solution apparently does not extend to our general case.

<u>Lemma 7</u>. <u>Assume</u> (3a) <u>and</u> (3b). <u>Let</u> u_t <u>be the minimal solution of</u> (1_t). <u>Then the first eigenvalue</u> μ_1 <u>of the eigenvalue problem</u>

(13) $$-\Delta v - g_s'(x,u_t)v = \mu v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

is > 0.

Proof. We follow Dancer [6] in this argument. Suppose by contradiction that $\mu_1 < 0$ and let $v > 0$ be the corresponding eigenfunction normalized by $\int v^2 = 1$. Look at $u_t + \delta v$ and write

$$-\Delta(u_t + \delta v) = g(x,u_t) + t\phi + h + \delta[g_s'(x,u_t)v + \mu_1 v]$$

and using the expression

$$g(x,u_t + \delta v) - g(x,u_t) = \int_0^1 g_s'(x,u_t + r\delta v)\delta v dr$$

we obtain

$$-\Delta(u_t + \delta v) = g(x,u_t + \delta v) + t\phi + h + \delta v\Big[\int_0^1 [g_s'(x,u_t) - g_s'(x,u_t + r\delta v)]dr + \mu_1\Big]$$

By the continuity of g_s' the expression inside the brackets is < 0 for $|\delta|$ small. So if we take $\delta < 0$ and small we get

$$-\Delta(u_t + \delta v) > g(x,u_t + \delta v) + t\phi + h$$

which says that $u_t + \delta v$ is a supersolution of (1_t). By Lemma 4 and using the monotone iteration method we would obtain a solution u of (1_t) with $u < u_t + \delta v < u_t$ in Ω, contradicting Corollary 6. □

For future purposes we need $\mu_1 > 0$. We can prove that this is so under a slightly stronger assumption than the one in Lemma 7.

Lemma 8. Assume (4a) *and* (3b). *Given* $h \in W^1$, *there is a* $t_2 \in \mathbb{R}$ *such that the first eigenvalue* μ_1 *of problem* (13) *for* $t < t_2$ *is positive*.

Proof. Let us assume by contradiction that there are $t_n \to -\infty$ such that the first eigenvalue of (13) with $t = t_n$ is 0. So there is $v_{t_n} > 0$ in Ω, $\int v_{t_n}^2 = 1$ such that

$$-\Delta v_{t_n} = g_s'(x,u_{t_n})v_{t_n} .$$

To simplify notation let us denote $v_n = v_{t_n}$, $u_n = u_{t_n}$. Then

$$\int |\nabla v_n|^2 = \int g_s'(x,u_n)v_n^2 \le C \int v_n^2 = C$$

where $C = \sup\{g_s'(x,s);\ x \in \bar{\Omega},\ s \le 0\}$ which exists in view of assumption (4a). So we may assume that $v_n \rightharpoonup v_0$ in H_0^1, $v_n \to v_0$ in L^2 and a.e., and that $v_n(x) \le h(x)$ for some $h \in L^2$. Clearly $\|v_0\|_{L^2} = 1$. Then

$$(14) \qquad -\int \Delta v_n \cdot \phi = \int g_s'(x,u_n)v_n\phi = \int_{u_n<s_0} + \int_{u_n>s_0}$$

where $s_0 < 0$ and $\underline{\mu} < \lambda_1$ (used next) are such that

$$g_s'(x,s) \le \underline{\mu} \quad \text{for} \quad x \in \bar{\Omega}, \quad s \le s_0 .$$

From (14) we obtain

$$(15) \qquad \lambda_1 \int v_n\phi \le \underline{\mu} \int v_n\phi + \int g_s'(x,u_n)v_n\phi\chi_{[u_n>s_0]}$$

Now we remark that a simple modification of Lemma 5 yields the following stronger conclusion: given any $\beta < 0$ there is a t_β such that problem (1_t) for $t < t_\beta$ has a supersolution $W \le \beta\phi$. This implies that $\chi_{[u_n>s_0]}$ tends to 0 a.e.. By Fatou's lemma we obtain from (15)

$$\lambda_1 \int v_0\phi \le \underline{\mu} \int v_0\phi$$

which is impossible. □

<u>Remark</u>. It follows from Lemma 8 that

$$\int |\nabla v|^2 - \int g_s'(x,u_t)v^2 \ge \mu_1 \int v^2 , \quad \mu_1 > 0$$

On the other hand we know that there are constants $c, c_0 > 0$ such that

$$\int |\nabla v|^2 - \int g_s'(x,u_t)v^2 \ge c \int |\nabla v|^2 - c_0 \int v^2$$

which is Garding's inequality. Using the previous inequality we can easily see that $c_0 = 0$.

2. **EXISTENCE OF THE SECOND SOLUTION.** Now fix $t < t_2$ and let w be the subsolution associated with problem (1_t), by Lemma 4. We now modify our function g as follows. Let $\tilde{g}(x,s)$ be the new function such that:

(i) $\tilde{g}(x,s) = g(x,s)$ for $x \in \Omega$, $s > w(x)$,

(ii) $\tilde{g}(x,s) > \underline{\mu}s - C$ and $\tilde{g}(x,s) > \bar{\mu}s - C$ for $x \in \Omega$ and $s \in \mathbb{R}$, where C, $\underline{\mu}$ and $\bar{\mu}$ are the constants in (7).

(iii) $\tilde{g}(x,s) > 0$ for $x \in \Omega$ and all s smaller than a certain constant s_1.

(iv) $\tilde{g}(x,s)$ has linear growth at $-\infty$, i.e., $\tilde{g}(x,s)$ is like cs, $c < 0$, at $-\infty$.

(v) $\tilde{g}(x,s)$ is a C^1 function.

In view of (ii) we see that all eventual solutions of the modified problem (1_t) with g replaced by $\tilde{g}$ are bounded from below by $w(x)$, see Lemma 4. So the solution of the modified problem are the same as the ones of the original problem. Thus from now on we assume that g has properties (i)-(v) above. Let us now look for solutions of (1_t) as critical points of the functional

$$J(u) = \frac{1}{2}\int |\nabla u|^2 - \int G(x,u) - \int fu$$

defined in H_0^1.

Lemma 9. *The functional* J *satisfies the Palais-Smale condition.*

Proof. Case 1. $N > 3$, $(N+1)/(N-1) < \sigma < (N+2)/(N-2)$. Let (u_n) be a sequence in H_0^1 such that $|J(u_n)| < C$ and $J'(u_n) \to 0$. We claim (u_n) contains a convergent subsequence in H_0^1. From $J'(u_n) \to 0$ we have for all $v \in H_0^1$

$$|\int \nabla u_n \nabla v - \int g(x,u_n)v - \int fv| < \varepsilon_n \|v\|_{H^1}, \quad \varepsilon_n \to 0 . \tag{16}$$

Let $s_2 < s_1$ be such that $G(x,s) < 0$ for $s < s_2$, and define

$$w_n = \begin{cases} u_n - s_2 & \text{if } u_n < s_2 \\ 0 & \text{if } u_n > s_2 \end{cases}$$

Taking $v = w_n$ *in* (16):

(17) $$\int |\nabla w_n|^2 - \int g(x,u_n)w_n \leq \int fw_n + \varepsilon_n \|w_n\|_{H^1}$$

and since $g(x,s) \geq 0$ for $s \leq s_2$ we obtain that $\|w_n\|_{H^1} \leq C$. It then follows from (17) that

(18) $$-\int_{u_n \leq s_2} g(x,u_n)u_n \leq C + C\|u_n\|_{H^1}$$

where we have used the fact that g has linear growth at $-\infty$. Next from $|J(u_n)| \leq C$ it follows

$$\int |\nabla u_n|^2 \leq C + 2\int G(x,u_n) + 2\int fu_n \leq C + 2\int_{u_n \geq s_3} G(x,u_n) + C\|u_n\|_{H^1}$$

where $s_3 > 0$ is chosen from (6) in such a way that

$$sg(x,s) - \theta G(x,s) \geq -\varepsilon s^2 g(x,s)^{2/N+1}, \quad x \in \bar{\Omega},\ s \geq s_3$$

for some given $\varepsilon > 0$. Then

(19) $$\int |\nabla u_n|^2 \leq C + C\|u_n\|_{H^1} + \frac{2}{\theta}\int_{u_n \geq s_3} u_n g(x,u_n) + \frac{2\varepsilon}{\theta}\int_{u_n \geq s_3} u_n^2 g(x,u_n)^{2/(N+1)}$$

Using (18) the first integral in the right side of (19) can be taken all over Ω, which is then estimated using (16) with $v = u_n$:

$$\int |\nabla u_n|^2 \leq C + C\|u_n\|_{H^1} + \frac{2}{\theta}\int |\nabla u_n|^2 + \frac{2\varepsilon}{\theta}\int_{u_n \geq s_3} u_n^2 g(x,u_n)^{2/(N+1)}$$

or

(20) $$\int |\nabla u_n|^2 \leq C + C\varepsilon\int_{u_n \geq s_3} u_n^2 g(x,u_n)^{2/(N+1)}$$

Now we claim that

(21) $$\int_{u_n \geq s_3} g(x,u_n)\phi \leq C$$

Indeed, it follows from (16) with $v = \phi$ that

$$-\varepsilon_n\|\phi\|_{H^1} \leq \lambda_1\int u_n\phi - \int g(x,u_n)\phi - \int f\phi \leq (\lambda_1 - \bar{\mu})\int u_n\phi + C$$

which implies that $\int u_n\phi$ and $\int g(x,u_n)\phi$ are bounded above. This gives (21) in view of hypothesis (iii) on g. The integral in (20) is estimated using Hölder's inequality by

$$\Big(\int_{u_n>s_3} g(x,u_n)\phi\Big)^{2/(N+1)}\Big(\int_{u_n>s_3} \phi^{-2/(N-1)}u_n^{2(N+1)/(N-1)}\Big)^{(N-1)/(N+1)}$$

which is then bounded by $c\|\nabla u_n\|^2_{L^2}$ using the Hardy-Sobolev inequality [7]. This shows that $\|\nabla u_n\|_{L^2}$ is uniformly bounded. So we may assume that $u_n \rightharpoonup u$ in H_0^1 and $u_n \to u$ in L^2. Now we claim that $u_n \to u$ in H_0^1. In the case $\sigma < (N+2)/(N-2)$ this follows readily from the fact that J' is an operator of the form identity minus a compact operator. Although this is not true in the case $\sigma = (N+2)/(N-2)$ we can still prove that $u_n \to u$ as follows. Let $\varepsilon > 0$ be given. Then there is an s_4 such that $g(x,s) < \varepsilon s^\sigma$ for $s > s_4$. Now using (16) with $v = u_n - u$ we have, with $C_1 > \|u_n\|_{H^1}, \|u\|_{H^1}$,

$$\int |\nabla(u_n-u)|^2 \le -\int \nabla u\cdot\nabla(u_n-u) + \int f(u_n-u) + 2C_1\varepsilon_n + \int g(x,u_n)(u_n-u)\ .$$

The first three terms in the above estimate go to zero as $n \to \infty$. We estimate the last term by

$$C\|u_n-u\|_{L^2} + \varepsilon\int_{u_n>s_4} u^\sigma|u_n-u|$$

Taking this last integral over the whole of Ω with $|u_n|$ replacing u_n, and using Hölder's inequality we have finally

$$\int |\nabla(u_n-u)|^2 \le -\int \nabla u\cdot\nabla(u_n-u) + C\|u_n-u\|_{L^2} + 2C_1\varepsilon_n + C_2\varepsilon$$

where C_2 is a constant independent of ε. This proves the claim.

Case 2. $N \ge 3$, $\sigma < (N+1)/(N-1)$. This case is much simpler. Taking (16) with $v = u_n$ and using property (iii) of g we obtain

$$(22)\qquad \int |\nabla u_n|^2 \le C + C\|u_n\|_{H^1} + \int_{u_n>s_4} g(x,u_n)u_n$$

The integral on the right side is then estimated using Hardy-Sobolev's inequality as above, leading to a uniform bound for the L^2-norm of the gradient of u_n. The convergence of a subsequence follows as in Case 1.

Case 3. $N = 2$. We proceed as in Case 2. □

Proof of Theorem 1 completed. In order to apply the mountain pass theorem it remains to check that

(i) there is $v \in H_0^1$ such that $J(v) < J(u_t)$,

(ii) there is an $r > 0$ such that $\inf\{J(u) : \|u - u_t\| = r\} > J(u_t)$

To see (i) we take $v = R\phi$:

$$J(R\phi) = \frac{1}{2}\lambda_1 R^2 - \int G(x,R\phi) - R\int f\phi$$

From (7), $G(x,s) \geq \frac{1}{2}\bar{\mu}s^2 - C$ and we obtain

$$J(R\phi) \leq \frac{1}{2}(\lambda_1 - \bar{\mu})R^2 - CR - C$$

which gives the claim by taking R sufficiently large.

As for (ii) we proceed as follows. Using the fact that u_t is a critical point of J we have:

$$J(u_t + v) - J(u_t) = \frac{1}{2}\int |\nabla v|^2 - \int [G(x,u_t + v) - G(x,u_t) - g(x,u_t)v] \tag{23}$$

By Taylor's formula, the expression into the brackets is $\frac{1}{2}g_s'(x,u_t)v^2 + r(x,v)$ where $|r(x,v)| \leq \varepsilon(x,v)|v|^2$ if $|v| < 1$ and $\varepsilon(x,v) \to 0$ as $v \to 0$. On the other hand $|r(x,v)| \leq c|v|^{\sigma+1}$ for $|v| > 1$. Altogether

$$|r(x,v)| \leq \varepsilon|v|^2 + c|v|^{\sigma+1}$$

which gives

$$\int |r(x,v)|dx \leq \varepsilon\|v\|^2_{L^2} + c\|v\|^{\sigma+1}_{L^{\sigma+1}} \leq \varepsilon\|v\|^2_{H^1} + c\|v\|^{\sigma+1}_{H^1}$$

Estimating (23) we obtain

$$J(u_t + v) - J(u_t) \geq \frac{1}{2}\int |\nabla v|^2 - \frac{1}{2}\int g_s'(x,u_t)v^2 - \varepsilon\|v\|^2_{H^1} - c\|v\|^{\sigma+1}_{H^1}$$

which in view of Lemma 8 (and the Remark right after it) is bounded below by $c\|v\|^2_{H^1} - c\|v\|^{\sigma+1}_{H^1}$. So if $\|v\|_{H^1} = r$ and r is small the quadratic term dominates and we have claim (ii). The final step is the regularity of this solution obtained by the mountain pass theorem. In the case $\sigma < (N + 2)/(N - 2)$ this follows readily via a standard bootstrap argument. In the case $\sigma = (N + 2)/(N - 2)$, this follows from Theorem 10 proved in the next section. □

3. A REGULARITY RESULT. In the present section we establish a regularity result for H_0^1 solutions of semilinear elliptic equations of second order. The growth of the nonlinearity $g(x,u)$ as $u \to \infty$ prevents a direct use of bootstrap arguments. The main ideas of the proof below are borrowed from Brézis-Kato [11] where they treat the linear Schrödinger equation.

Theorem 10. *Let* $g : \Omega \times R \to R$ *be a Carathéodory function such that*

$$|g(x,s)| \le c|s|^{\sigma} + c(x)$$

where $\sigma = (N + 2)/(N - 2)$, c *is a constant and* $c(x)$ *is an* $L^{2N/(N-2)}$ *function. If* $u \in H_0^1(\Omega)$ *is a solution of*

(24) $$-\Delta u = g(x,u) \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega$$

then $u \in L^p$ *for some* $p > 2N/(N - 2)$.

Remark. Once we know $u \in L^p$ for some $p > 2N/(N - 2)$ then a bootstrap argument shows that $u \in C^{1,\alpha}$. If more regularity of g is required (for example C^1 in the situation of the previous section) then $u \in C^{2,\alpha}$.

Proof. Let $\chi \in C_0^{\infty}(R)$ with compact support containing a neighborhood of zero. Then g decomposes as $g = g_1 + g_2$ where $g_1(x,s) = \chi(s)g(x,s)$ and $g_2(x,s) = (1 - \chi(s))g(x,s)$. Consequently $g(x,u(x))$ where $u \in H_0^1$ is the given solution of (24) can be written as

$$g(x,u(x)) = a(x)u(x) + b(x)$$

where $a(x) \in L^{N/2}$ and $b(x) \in L^{2N/(N-2)}$. Now we use the following lemma from Brézis-Kato [11, p. 139]: "Given $a(x) \in L^{N/2}$ and $\varepsilon > 0$ there is a constant $k_\varepsilon > 0$ such that

(25) $$\int |a(x)|u^2 \le \varepsilon \int |\nabla u|^2 + k_\varepsilon \int u^2, \quad \forall\, u \in H_0^1 ."$$

Let k_1 be the constant in (25) corresponding to $\varepsilon = 1$. It is easy to see that, for $k > k_1$, the problem

$$-\Delta v + kv = a(x)v \ \text{in} \ \Omega, \quad v = 0 \ \text{on} \ \partial\Omega$$

has only the trivial solution $v = 0$. So the problem

(26) $$-\Delta v + kv = a(x)v + b(x) + ku(x) \ \text{in} \ \Omega, \quad v = 0 \ \text{on} \ \partial\Omega$$

has a <u>unique</u> solution v, which happens to be $v = u$. We now aim at proving that the unique solution $v \in H_0^1$ of the problem

$$-\Delta v + kx = a(x)v + d(x) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

where $d(x)$ is an $L^{2N/(N-2)}$ function, is in some L^p with $p > 2N/(N-2)$. For that matter we truncate $a(x)$ as follows

$$a_\ell(x) = \begin{cases} -\ell, & \text{if } a(x) < -\ell \\ a(x), & \text{if } |a(x)| < \ell \\ \ell, & \text{if } a(x) > \ell \end{cases}$$

We observe that (25) holds with a replaced by a_ℓ and k_ε is independent of ℓ, $\ell \to +\infty$. The problem

(27) $$-\Delta v + kv = a_\ell(x)v + d(x) \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega$$

has also a unique solution $v_\ell \in H_0^1$. It follows from (25) that $\|v_\ell\|_{H^1} < C$ for all ℓ. We claim that $v_\ell \to v$ in H_0^1, as $\ell \to +\infty$. From (26) and (27) we obtain

$$\int |\nabla(v_\ell - v)|^2 + k \int (v_\ell - v)^2 < \int a_\ell(v_\ell - v)^2 + c\|a_\ell - a\|_{L^{N/2}}\|v_\ell - v\|_{L^{2N/(N-2)}}$$

and the result follows applying (25) with $\varepsilon > 0$ sufficiently small, using Sobolev embedding and the fact that $v_\ell \to v$ in L^2. Next we claim that the solution v_ℓ of (27) is in some L^p with $p > 2N/(N-2)$ and that

(28) $$\|v_\ell\|_{L^p} < \text{constant independent of } \ell .$$

We now truncate v_ℓ

$$v_{\ell,n}(x) = \begin{cases} -n, & v_\ell(x) < -n \\ v_\ell(x), & \text{if } |v_\ell(x)| < n \\ n, & \text{if } v_\ell(x) > n \end{cases}$$

obtaining functions in $H_0^1 \cap L^\infty$. To simplify our notation let us call $w = v_\ell$ and $w_n = v_{\ell n}$. The function $w_n|w_n|^{q-1}$, for any $q > 1$, is in H_0^1, and

$$\nabla(w_n|w_n|^{q-1}) = q|w_n|^{q-1}\nabla w_n .$$

Multiplying equation (27) by $w_n|w_n|^{q-1}$ and integrating by parts we obtain

$$(29)\qquad \frac{4q}{(q+1)^2}\int |\nabla(w_n|w_n|^{\frac{q-1}{2}})|^2 + k\int ww_n|w_n|^{q-1} = \int a_\ell w w_n|w_n|^{q-1} + \int dw_n|w_n|^{q-1}$$

The second integral in the left side of (29) is positive and will give us no problems. The second integral in the right side of (29) is estimated by

$$\left(\int |d|^{q+1}\right)^{1/(q+1)}\left(\int |w_n|^{q+1}\right)^{q/(q+1)}$$

which is bounded by $\|d\|_{L^{2N/(N-2)}}\cdot\|v_\ell\|_{L^{2N/(N-2)}}^{(N+2)/(N-2)}$, taking (and we shall assume from now on) $q+1 = 2N/(N-2)$, and consequently bounded independent of ℓ. We now estimate the first integral in the right side of (29) by

$$(30)\qquad \int_{|w|\geq n} + \int_{|w|\leq n} \leq \ell \int_{|w|\geq n} |w|^{q+1} + \int |a_\ell||w_n|^{q+1}$$

The second integral in the right side of (30) is estimated using (25) by

$$\varepsilon\int |\nabla(w_n|w_n|^{\frac{q-1}{2}})|^2 + k_\varepsilon \int |w_n|^{q+1}$$

All this information used in (29) leads to

$$\int |\nabla(w_n|w_n|^{\frac{q-1}{2}})|^2 \leq C + \ell\int_{|w|\geq n} |w|^{q+1}$$

where C is independent of ℓ and n. Making $n \to +\infty$, and using Sobolev embedding we have

$$(31)\qquad \limsup_{n\to\infty} \int |w_n|^{(q+1)\frac{N}{N-2}} \leq C\ .$$

Since $w_n \to w$ in $L^{2N/(N-2)}$, it follows from (31), using reflexivity of L^p spaces that $w \in L^{(q+1)N/(N-2)}$. So (28) is proved with $p = (q+1)N/(N-2)$ and $q+1 = 2N/(N-2)$. Finally to prove that $v \in L^p$ we use the same argument as above, since $v_\ell \to v$ in $L^{2N/(N-2)}$ and the p-norms of v_ℓ are uniformly bounded. □

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This paper establishes the existence of two solutions for some problems of the Ambrosetti-Prodi type. The following result is a sample of the results. Let $g : R \to R$ be a C^1-function such that $\lim_{s\to-\infty} g'(s) < \lambda_1 < \lim_{s\to+\infty} g'(s)$, where λ_1 is the first eigenvalue of the Laplacian with Dirichlet boundary conditions, and the limits could be

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ABSTRACT (cont.)

infinite. Suppose that g behaves at $+\infty$ like u^p, $1 < p < (N+2)/(N-2)$ or even like $u^{(N+2)/(N-2)}/\ln u$, where $N \geq 3$. Let Ω be a bounded smooth domain in R^N and let ϕ be the first eigenfunction corresponding to λ_1, which is > 0 in Ω. Then given any $h \in C^{\alpha}(\bar{\Omega})$ such that $\int h\phi = 0$, there is $t_0 \in R$ such that the Dirichlet problem

$$-\Delta u = g(u) + t\phi + h, \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

has at least two $C^{2,\alpha}$ solutions for each $t < t_0$. The author uses the method of monotone iterations to obtain the first solution and a variational argument to get the second. The variational solution is subsequently regularized.

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